Application of Integrals

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1. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis in the first quadrant.

   ### Solutions :

   The area of the region bounded by the curve, $y^2 = x$, the lines, $x = 1$ and $x = 4$, and the x-axis is the area $ABCDA$. Area $ABCDA = \int_1^4 \sqrt{x}dx$

   $= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$

   $= \frac{2}{3}[(4)^{\frac{3}{2}} - (1)^{\frac{3}{2}}]$

   $= \frac{2}{3}[8 - 1]$

   $= \frac{14}{3} sq.units$

2. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.

   ### Solutions :

   The area of the region bounded by the curve, $y^2 = 9x, x = 2$, and $x = 4$, and the x-axis is the area $ABCDA$.

   Area $ABCDA = \int_2^4 3\sqrt{x}\mathrm{d}x$

   $= 3[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}]_2^4$

   $= 2[x^{\frac{3}{2}}]_2^4$

   $= 2[4^{\frac{3}{2}} - 2^{\frac{3}{2}}]$

   $= 2[2^3 - 8^{\frac{1}{2}}] = 2[8 - 2\sqrt{2}]$

   $= 16 - 4\sqrt{2} sq.units$

3. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y-axis in the first quadrant.

   ### Solutions :

   The area of the region bounded by the curve, $x^2 = 4y, y = 2$, and $y = 4$, and the y-axis is the area $ABCDA$.
   Area of $ABCDA = \int_2^4 x\mathrm{d}y$

   $x^2 = 4y$

   $x = 2\sqrt{y}$

   $\int_2^4 x\mathrm{d}y = \int_2^4 2\sqrt{y}\mathrm{d}y$

   $= 2\int_2^4 \sqrt{y}\mathrm{d}y$

   $= 2\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$

   $= \frac{4}{3}[(4)^{\frac{3}{2}} - (2)^{\frac{3}{2}}]$

   $= \frac{4}{3}[8 - 2\sqrt{2}]$

   $= (\frac{32 - 8\sqrt{2}}{3}) sq.units$

4. Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

   ### Solutions :

   The given equation of the ellipse,$\frac{x^2}{16} + \frac{y^2}{9} = 1$can be represented as
   It can be observed that the ellipse is symmetrical about x-axis and y-axis.
   $\therefore$ Area bounded by ellipse =4* Area of $OABO$
   Area $OABO = \int_0^4 y\mathrm{d}x$

   $\frac{x^2}{16} + \frac{y^2}{9} = 1$

   $\implies \frac{y^2}{9} = 1 - \frac{x^2}{16}$

   $\implies y^2 - 9(1 - \frac{x^2}{16})$

   $y = 3\sqrt{1 - \frac{x^2}{16}}$

   Area $OABO = \int_0^4 3\sqrt{1 - \frac{x^2}{16}\mathrm{d}x}$

   $= \frac{3}{4}\int_0^4 \sqrt{16 - x^2}\mathrm{d}x$

Substitute $x = 4\sin\theta, \theta = \sin^{-}1\dfrac{x}{4}$
$= \dfrac{3}{4}\int_0^{\frac{\pi}{2}} \sqrt{16-16\sin^2\theta}.4\cos\theta \mathrm{d}\theta$
$= 12\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}, \cos\theta \mathrm{d}\theta$
$= 12\int_0^{\frac{\pi}{2}} \cos^2\theta \mathrm{d}\theta$
$= 12\int_0^{\frac{\pi}{2}} \dfrac{1+\cos 2\theta}{2}\mathrm{d}\theta$
$= 6\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\mathrm{d}\theta$
$= 6[\theta + \dfrac{\sin 2\theta}{2}]_0^{\frac{\pi}{2}}$
$= 6[\dfrac{\pi}{2} + \dfrac{\sin\pi}{2} - 0 - \dfrac{\sin 0}{2}]$
$= 6[\dfrac{\pi}{2}] = 3\pi$

Therefore, area bounded by the ellipse $= 4 * 3\pi = 12\pi sq.units$

5. Find the area of the region bounded by the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

Solutions :

The given equation of the ellipse can be represented as
$\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$
$\dfrac{y^2}{9} = 1 - \dfrac{x^2}{4}$
$y^2 = 9(1 - \dfrac{x^2}{4})$
$\implies y = 3\sqrt{1-\dfrac{x^2}{4}}$......(1)
It can be observed that the ellipse is symmetrical about x-axis and y-axis.
$\therefore$ Area bounded by ellipse = 4 *Area OABO

Area of $OABO = \int_0^2 y\mathrm{d}x$
$= \int_0^2 3\sqrt{1-\dfrac{x^2}{4}}\mathrm{d}x$ [Using (1)]
$= \dfrac{3}{2}\int_0^2 \sqrt{4-x^2}\mathrm{d}x$
Substitute $x = 2\sin\theta \implies \theta = \sin^{-}1(\dfrac{x}{2})$
$\mathrm{d}x = 2\cos\theta\mathrm{d}\theta$
when,$x = 0, \theta = 0\& x = 2\ \ \theta = \dfrac{x}{2}$
$\therefore \dfrac{3}{2}\int_0^2 \sqrt{4-x^2}\mathrm{d}x$
$= \dfrac{3}{2}\int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2\theta}.2\cos\theta\mathrm{d}\theta$
$=3\int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2\theta}.\cos\theta\mathrm{d}\theta$
$= 6\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cos\theta\mathrm{d}\theta$
$= 6\int_0^{\frac{\pi}{2}} \cos 2\theta\mathrm{d}\theta$
$= 6\int_0^{\frac{\pi}{2}} \dfrac{1+\cos 2\theta}{2}\mathrm{d}\theta$
$= \dfrac{6}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\mathrm{d}\theta$
$= 3[0 + \dfrac{\sin 2\theta}{2}]_0^{\frac{\pi}{2}}$
$3[\dfrac{\pi}{2} + \dfrac{\sin\pi}{2} - 0]$
$= 3 * \dfrac{\pi}{2} = \dfrac{3\pi}{2}$
Therefore, area bounded by the ellipse$= 4 * \dfrac{3\pi}{2} = 6\pi sq.units$

6. Find the area of the region in the first quadrant enclosed by x-axis, line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$

Solutions :

The area of the region bounded by the circle, $x^2 + y62 = 4, x = \sqrt{3}y$, and the x-axis is the area OAB
Substituting $x = \sqrt{3}y$ in $x^2 + y^2 = 4$ for finding the point of intersection.
$\therefore (\sqrt{3}y) + y^2 = 4 \Rightarrow y^2 - 1 \Rightarrow y = \pm 1, x = \pm\sqrt{3}$
The point of intersection of the line and the circle in the first quadrant is$(\sqrt{3}, 1)$
Area $OABO$=Area ΔOCA + Area $ACBA$
Area of $OAC = \dfrac{1}{2} * OC * AC = \dfrac{1}{2} * \sqrt{3} * 1 = \dfrac{\sqrt{3}}{2}$ ...(1)
Area of$ABCA = \int_{\sqrt{3}}^2 y\mathrm{d}x$
$\int_{\sqrt{3}}^2 y\mathrm{d}x = \int_{\sqrt{3}}^2 \sqrt{4-x^2}\mathrm{d}x$
$x = 2\sin\theta \quad \theta = \sin^{-1}(\dfrac{x}{2})$

when $x = 2 \quad \theta = \dfrac{\pi}{2}$
$x = \sqrt{3} \ \ \theta = \dfrac{\pi}{3}$

$\therefore \int_{\sqrt{3}}^{2} \sqrt{4-x}\mathrm{d}x$
$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sqrt{4-4\sin^2\theta}(2\cos\theta)\mathrm{d}\theta$
$= 4\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2\theta\mathrm{d}\theta$
$= 4\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 1+\dfrac{\cos 2\theta}{2}\mathrm{d}\theta$
$= 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1+\cos 2\theta)\theta$
$= 2[\theta+\dfrac{\sin 2\theta}{2}]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$
$= 2[\dfrac{\pi}{2}+\dfrac{1}{2}\sin\pi-\dfrac{\pi}{3}-\dfrac{1}{2}\sin\dfrac{2\pi}{3}]$
$= 2[\dfrac{\pi}{2}-\dfrac{\pi}{3}-\dfrac{1}{2}*\dfrac{\sqrt{3}}{2}] = 2[\dfrac{\pi}{6}-\dfrac{\sqrt{3}}{4}] = \dfrac{\pi}{3}$ ......(2)
From (1) amp; (2)
Area of $OAB = \dfrac{\sqrt{3}}{2}+2[\dfrac{\pi}{6}-\dfrac{\sqrt{3}}{4}] = \dfrac{\pi}{3}$
Therefore, area enclosed by x-axis, the line $x=\sqrt{3}y$,and the circle $x^2+y^2=4$ in the first quadrant $=\dfrac{\pi}{3}$ sq.units

7. Find the area of the smaller part of the circle $4x^2+y^2=a^2$ cut off by the line $x=\dfrac{a}{\sqrt{2}}$.

Solutions :

The area of the smaller part of the circle $4x^2+y^2=a^2$ cut off by the line $x=\dfrac{a}{\sqrt{2}}$, is the area $ABCDA$
It can be observed that the area $ABCDA$ is symmetrical about x-axis.
Area of $ABCA = \int_{\frac{a}{\sqrt{2}}}^{a} y\mathrm{d}x$
$= \int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{a^2-x^2}\mathrm{d}x$
$x = a\sin\theta \quad \mathrm{d}x = a\cos\theta\mathrm{d}\theta$
$x = \dfrac{a}{\sqrt{2}} \quad \theta = \sin^{-1}(\dfrac{x}{a})$
$= \sin^{-1}(\dfrac{1}{\sqrt{2}}) = \dfrac{\pi}{4}$
$x = a, \theta = \sin^{-1}(\dfrac{a}{a}) = \dfrac{\pi}{2}$
$\therefore = \int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{a^2-x^2}\mathrm{d}x$
$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{a^2-a^2\sin^2\theta}.(a\cos\theta)\mathrm{d}\theta$
$\Rightarrow a^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2\theta$
$= a^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \dfrac{1+\cos 2\theta}{2}\mathrm{d}\theta$
$= \dfrac{a^2}{2}[\theta+\dfrac{\sin 2\theta}{2}]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$
$= \dfrac{a^2}{2}[\dfrac{\pi}{2}+\dfrac{\sin\pi}{2}-\dfrac{\pi}{4}-\dfrac{\sin\frac{\pi}{2}}{2}]$
‘ $= \dfrac{a^2}{2}[\dfrac{\pi}{4}-\dfrac{1}{2}]$
$= \dfrac{a^2}{4}[\dfrac{\pi}{2}-1]$
$\Rightarrow ABCD = 2[\dfrac{a^2}{4}(\dfrac{\pi}{2}-1)]$
$= \dfrac{a^2}{2}(\dfrac{\pi}{2}-1)$
Therefore, the area of smaller part of the circle,$x^2+y^2=a62$,cut off by the line $x=\dfrac{a}{\sqrt{2}}$,is $\dfrac{a^2}{2}(\dfrac{\pi}{2}-1)$sq. units.

8. The area between $x=y^2$ and $x=4$ is divided into two equal parts by the line $x=a$, find the value of a.

Solutions :

The line $x=a$, divides the area bounded by the parabola $x=y^2$ and $x=4$ into two equal parts.
$\therefore$ Area $OADO$ = Area $ABCDA$

It can be observed that the given area is symmetrical about x-axis.
Area of $OEDO = \dfrac{1}{2}$ Area of $OADO$
Area of $EFCDE = \dfrac{1}{2}$Area of $ABCDA$
Therefore, Area $OEDO$ = Area $EFCDE$
Area $OEDO = \int_0^a y\mathrm{d}x$
$= \int_0^a \sqrt{x}\mathrm{d}x$
$\left[\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a$
$= \dfrac{2}{3}(a)^{\frac{3}{2}}$ (1)
Area of $EFCDE = \int_a^4 y\mathrm{d}x = \int_a^4 \sqrt{x}\mathrm{d}x$

$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]$
$= \frac{2}{3}[4^{\frac{3}{2}} - a^{\frac{3}{2}}]$
$= \frac{2}{3}[8 - a^{\frac{3}{2}}] \quad ....(2)$

From (1) and (2), we obtain
$\frac{2}{3}(a)^{\frac{3}{2}} = \frac{2}{3}[8 - (a)^{\frac{3}{2}}]$
$\Rightarrow 2.(a)^{\frac{3}{2}} = 8$
$\Rightarrow (a)^{\frac{3}{2}} = 4$
$\Rightarrow a = (4)^{\frac{2}{3}}$
Therefore, the value of a is $(4)^{\frac{2}{3}}$

9. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

Solutions :

The area bounded by the parabola, $x^2 = y$, and the line , $y = |x|$, can be represented as
The given area is symmetrical about y-axis.
$\therefore$ Area $OACO$ = Area $ODBO$
The point of intersection of parabola $x^2 = y$ and line $y = x$ is $A(1, 1)$.
Area of $OACO$ = Area $\Delta OAM-$ Area $OMACO$
$\therefore$ Area of $\Delta OAM = \frac{1}{2} * OM * AM = \frac{1}{2} * 1 * 1 = \frac{1}{2}$
Area of $OMACO = \int_0^1 y\mathrm{d}x = \int_0^1 x^2\mathrm{d}x = [\frac{x^3}{3}]_0^1 = \frac{1}{3}$
$\Rightarrow$ Area of $OACO$ = Area of $\Delta OAM-$ Area of $OMACO$
$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$
Therefore, required area $= 2[\frac{1}{6}] = \frac{1}{3}$ sq.units.

10. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

Solutions :

The area bounded by the curve, $x^2 = 4y$, and line, $x = 4y - 2$, is represented by the shaded area $OBAO$.
Let A and B be the points of intersection of the line and parabola.
Substituting $x = 4y - 2$ in $x^2 = 4y$
$(4y - 2)^2 = 4y$
$16y^2 - 16y + 4 = 4y$
$16y^2 - 20y + 4 = 0$
$4y^2 - 5y + 1 = 0$
$(4y - 1)(y - 1) = 0$
$y = \frac{1}{4}, x = -1$
$y = 1, x = 2$
Coordinates of point A are $(-1, \frac{1}{4})$
Coordinates of point B are $(2, 1)$.
We draw AL and BM perpendicular to x-axis.
It can be observed that,
Area $OBAO$ = Area $OBCO$+Area $OACO$ ...(1)
Area $OMBCO$ = Area under the line $x = 4y - 2$ between $x = 0$ and $x = 2$
Area $OMBCO = \int_0^2 \frac{x+2}{4}\mathrm{d}x$
Area $OMBO$ = Area under the curve $x^2 = 4y$ between $x = 0$ and $x = 2$
Area $OMBO = \int_0^2 \frac{x^2}{4}\mathrm{d}x$
Then, Area $OBCO$ = Area $OMBCO-$ Area $OMBO$
$= \int_0^2 \frac{x+2}{4}\mathrm{d}x - \int_0^2 \frac{x^2}{4}\mathrm{d}x$
$= \frac{1}{4}[\frac{x^2}{2} + 2x]_0^2 - \frac{1}{4}[\frac{x^3}{3}]_0^2$
$= \frac{1}{4}[2 + 4] - \frac{1}{4}[\frac{8}{3}]$
$= \frac{3}{2} - \frac{2}{3} = \frac{5}{6}$
Area $OLACO$ = Area under the line $x = 4y - 2$ between $x = -1$ and $x = 0$
Area $OLACO = \int_{-1}^0 \frac{x+2}{4}\mathrm{d}x$
Area $OLACO$= Area under the curve $x^2 = 4y$ between $x = -1$ and $x =$ Area $OMBO = \int_0^2 \frac{x^2}{4}\mathrm{d}x$
Area $OACO$ = Area $OLACO-$ Area $OLAO$
$\int_{-1}^0 \frac{x+2}{4}\mathrm{d}x - \int_{-1}^0 \frac{x^2}{4}\mathrm{d}x$
$= \frac{1}{4}[\frac{x^2}{2} + 2x]_{-1}^0 - \frac{1}{4}[\frac{x^3}{3}]_{-1}^0$
$= \frac{1}{4}[\frac{0}{2} + 0 - \frac{(-1)^2}{2} - 2(-1)] - \frac{1}{4}[\frac{0^3}{0} - \frac{(-1)^3}{3}]$
$= -\frac{1}{4}[\frac{(-1)}{2} + 2(-1)] - [-\frac{1}{4}(\frac{(-1)^3}{3})]$

$= -\frac{1}{4}[\frac{1}{2} - 2] - \frac{1}{12}$
$= \frac{1}{2} - \frac{1}{8} - \frac{1}{12}$
$= \frac{7}{24}$
Therefore, required area $= (\frac{5}{6} + \frac{7}{24}) = \frac{9}{8}$ sq.units